**Black Scholes Option Pricing Model**

**Normal Distribution**

# Normal Distribution

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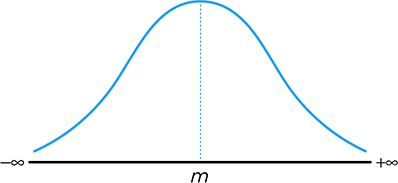
$$X \thicksim N(\mu, \sigma^2)$$



* Special distribution where the **parameters are the mean and variances themselves**

$$\text{f}(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2}$$





## Linear Combination of Normal Distributions

* Given two independently distributed Normal Variables, any **linear combination** of the two will be normally distributed as well
* Note that the variance will ALWAYS increase while the mean will be scaled accordingly

$$X \thicksim N(\mu\_x, \sigma^2\_x)$$

$$Y \thicksim N(\mu\_y, \sigma^2\_y)$$

$$X \pm Y \thicksim N(\mu\_x \pm \mu\_y, \sigma^2\_x + \sigma^2\_y)$$







# Standard Normal Distribution

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$$Z \ thicksim N(0, 1)$$



We can convert regular Normal Variables into Standard Normal Variables via **Normalization**

$$Z = \frac{X - \mu}{\sigma}$$



$$P(X < c)$$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{c - \mu}{\sigma}\right)$$

$$= P\left(Z < \frac{c - \mu}{\sigma}\right)$$







Note that if there is a constant,

$$P(X + k < c)$$

$$= P\left(\frac{x + k - \mu}{\sigma} < \frac{c - \mu}{\sigma}\right)$$

$$= P\left(\frac{X-\mu}{\sigma} < -\frac{k}{\sigma} + \frac{c - \mu}{\sigma}\right)$$

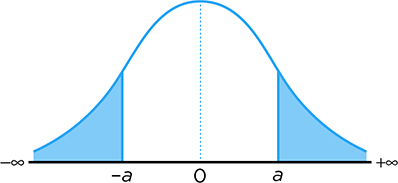






## Standard Normal Probabilities

Probability of Z < a 
Area under the curve before a 
< a) = N(a) 
Probability ofZ > a 
Area under the curve after a 





$$P(Z > a) = P(Z < -a)$$

$$P(Z < -a) = 1 – P(Z < a) = 1 – N(a)$$

$$P(a\_1 < Z < a\_2) = P(Z < a\_2) – P(Z < a\_1)$$\

$$N(a) = \int^a\_{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}a^2} \mathrm{d}a$$









## Standard Normal Table

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## Algebraic Manipulation

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**Log Normal Distribution**

# Log-Normal Distribution

* If the **natural logarithm of a variable is normally distributed**, then the original variable itself is **Log Normally distributed**

$$\ln{Y} \thicksim N(\mu, \sigma^2)$$

$$Y \thicksim \text{Lognormal}(\mu, \sigma^2)$$





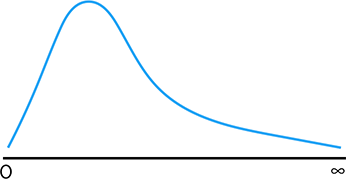
* + 
  + They are NOT the Mean and Variance of the Log Normal Distribution

$Y = e^X$ where $X$ is normally distributed

$$f(y) = \frac{1}{x\sigma\sqrt{2\pi}e^{-\left(\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right)}$$







## Properties of the Log Normal Distribution

* We can use the **Moment Generating Function** of the Normal Distribution to derive the Moments for the Log Normal Distribution
* Log normal distributions are ALWAYS positive
* The **PRODUCT** of two Log Normal Variables will still be Log Normally Distributed

$$M\_X(t) = E(e^{Xt}) = e^{\mu t + \frac{1}{2}\sigma^2t^2}$$



|  |  |
| --- | --- |
| **First Moment** | **Second Moment** |
|  |  |

# Normally Distributed Returns

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* The return is made of two components – **Dividend Yield and Capital Gain**

$$S\_0 \cdot e^r = e^{qt}S\_t$$

$$e\_r = e^{qt}\frac{S\_t}{S\_0}$$

$$r = \ln\left(e^{qt} \frac{S\_t}{S\_0}\right)$$

$$r = qt + \ln\frac{S\_t}{S\_0}$$











We assume that **continuous returns** are **normally distributed**

* This is because we can break the return up into the **sum of many smaller returns**
* If returns in **non-overlapping time periods** are **i.i.d.**, then the sum of these returns (Continuous return) is **approximately normal** via the **Central Limit Theorem**
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  + 

$$\therefore R \thicksim N(\mu t, \sigma^2 t)$$



Table

Description automatically generated with low confidence

# Log-Normal Stock Prices

Since the dividend yield is a **constant**,

$$r – qt \thicksim N(\mu t, \sigma^2t)$$

$$\ln\frac{S\_t}{S\_0} \ thicksim N(\mu t, \sigma^2t)$$

$$\therefore \frac{S\_t}{S\_0} \ thicksim \text{Lognormal}(\mu t, \sigma^2t)$$







Expressed differently,

$$\ln\frac{S\_t}{S\_0} = (r-q)t$$

$$\frac{S\_t}{S\_0} = e^{(r-q)t}$$

$$S\_t = S\_0e^{(r-q)t} \thicksim \text{Lognormal}(\mu t, \sigma^2t)$$

$\therefore$ \*\*Stock Prices are Log Normally Distributed\*\*









We **consider the first moment** of the Log normal distribution:

$$E[S\_t] = S\_0 \cdot E[e^{(r-q)t}]$$

$$E[S\_t] + S\_0 \cdot e^{\mu t + \frac{1}{2}\sigma^2t}$$







$$S\_0 \cdot e^{\alpha T} = e^{qT} \cdot E(S\_t)$$

$$E(S\_t) = S\_0 \cdot e^{(\alpha – q)t}$$





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Comparing both equations,

$$\mu + \frac{1}{2}\sigma^2 = \alpha - q$$

$$\mu = \alpha – q - \frac{1}{2}\sigma^2$$





We obtain an expression for the **log-normally distributed future stock prices:**

$$\therefore \ln\frac{S\_t}{S\_0} \thicksim N(\mu T, \sigma^2T)$$

$$\ln S\_T - \ln S\_0 \thicksim N(\mu T, \sigma^2T)$$

$$\ln S\_t \thicksim N(\ln S\_0 + \mu t, \sigma^2t)$$

$$\ln S\_t \thicksim N[\ln S\_0 + \left(\alpha – q - \frac{1}{2}\sigma^2\right)t, \sigma^2t]$$

$$R \thicksim N\left(\alpha – q - \frac{1}{2}\sigma^2, \sigma^2t\right)$$











We obtain an expression for the first two moments of a log-normal distribution:

$$E(S\_t) = e^{\ln S\_0 + \left(\alpha – q - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2t\right)} = S\_0e^{(\alpha-q)}t$$

$$\mathrm{Var}(S\_t) = [E(S\_t)]^2 \left(e^{\sigma^2t} - 1\right)$$





Skipping the formal proof, we can also obtain the **Conditional Expectation**:

$$E(S\_T \mid S\_T > K) = E(S\_T) \cdot \frac{N(\hat{d\_1})}{N(\hat{d\_2})}$$



* 
* The logic can be understood as follows:
  + 
  + 
  + 

Similarly, we obtain an expression for the actual stock price at a given time,

$$\therefore S\_t = S\_0e^{\left(\alpha – q - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t} \cdot Z}$$



* Note that this formula can be used to determine the stock price at **any period at a given percentile**
* Note that we need to convert the percentile back into the **Z value**

## Using Historical Data

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* We can follow the three step process to determine the parameters from historical data:

Most likely, the data will be given in **monthly format**. We will first need to convert the data into a **continuously compounded format**,

$$R\_\text{Continuous} = \ln\frac{S(t)}{S(t-1)}$$

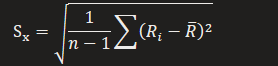


Next, we can calculate the Sample Mean and Sample Standard Deviation using the **Calculator Function**,

$$\bar{X} = \frac{1}{n}(R\_1 + R\_2 + \dots + R\_n) = \frac{1}{n}\Sigma{R\_i}$$

$$S\_X = \sqrt{\frac{1}{n-1}\Sigma{(R\_i - \bar{R})^2}}$$





Lastly, we convert them to **ANNUAL format**,

$$S^2\_{x, \text{Annual}} \cdot h = S^2\_{x, h}$$

$$\left(\bar{x}\_{\text{Annual}} – q - \frac{1}{2}S^2\_{x, \text{Annual}}\right) \cdot h = \bar{x}\_h$$





## Relation to Forward Prices

* Logically speaking, we would only enter a forward contract if we **expect** a positive payoff from it
* Thus, Expected Stock Price should always be **larger than the Expected Stock Price**

$$F\_0 = S\_0e^{(r-q)t}$$

$$E(S\_T) = S\_0e^{\alpha – q}t$$







$$E(S\_T) > F\_0$$



**Black Scholes Formula**

# Option Pricing with Lognormal Stocks

|  |  |
| --- | --- |
| **Call Option** | **Put Options** |
|  |  |
| Since the **second term is always 0**, we omit it from the equation | Since the **second term is always 0**, we omit it from the equation |
| We first consider an expression for the probability: | We first consider an expression for the probability: |
| We consider an expression for the expectation (Formal proof skipped): | We consider an expression for the expectation (Formal proof skipped): |
| Combining the above terms, | Combining the above terms, |
| The price of the option is thus the **PV of the expected payoff:** | The price of the option is thus the **PV of the expected payoff:** |

# Black Scholes Formula

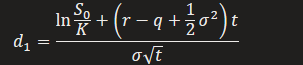
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  + These values are **not known beforehand** – so they are **purely theoretical**
* If we assume that the investor is **risk neutral –** it means that the investor **does not consider risk** in evaluating their investment decisions
  + 

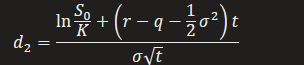
Under the **Risk Neutral Assumption**,

$$d\_1 = \frac{\ln\frac{S\_0}{K} + \left(r – q + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

$$d\_2 = \frac{\ln\frac{S\_0}{K} + \left(r – q - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

$$d\_2 = d\_2 - \sigma\sqrt{t}$$







|  |  |
| --- | --- |
| **Call Option** | **Put Option** |
|  |  |

More generally, we can express it in the form of **Prepaid Forward Prices:**

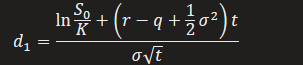
$$d\_1 = \frac{\ln\frac{S\_0}{K} + \left(r – q + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

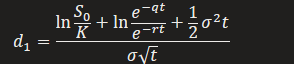
$$d\_1 = \frac{\ln\frac{S\_0}{K} + \ln\frac{e^{-qt}}{e^{-rt}} + \frac{1}{2}\sigma^2t}{\sigma\sqrt{t}}$$

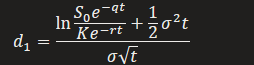
$$d\_1 = \frac{\ln\frac{S\_0e^{-qt}}{Ke^{-rt}} + \frac{1}{2}\sigma^2t}{\sigma\sqrt{t}}$$

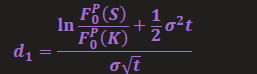
$$d\_1 = \frac{\ln\frac{F^P\_0(S)}{F^P\_0(K)} + \frac{1}{2}\sigma^2t}{\sigma\sqrt{t}}$$

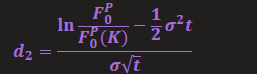
$$d\_2 = \frac{\ln\frac{F^P\_0}{F^P\_0(K)}- \frac{1}{2}\sigma^2t}{\sigma\sqrt{t}}$$











|  |  |
| --- | --- |
| **Call Option** | **Put Option** |
|  |  |



$$\sigma^2 = \frac{\mathrm{Var}(\ln F^P\_0)}{t}$$

$$\sigma^2 = \frac{\mathrm{Var}(\ln (F\_0 \cdot e^{-rt}))}{t}$$

$$\sigma^2 = \frac{\mathrm{Var}(\ln F\_0 \cdot -rt}{t}$$









## Black Scholes Special Cases

* There are two unique scenarios in which the Black Scholes formula will be simplified - allowing for us to **solve for a specific input**

|  |  |  |
| --- | --- | --- |
| **At the Money** | **Equal Rates** | **Both Scenarios** |
|  |  |  |
|  |  |  |